

# Some physical and chemical indices of clique-inserted-lattices

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## Abstract

The operation of replacing every vertex of an  $r$ -regular lattice  $H$  by a complete graph of order  $r$  is called clique-inserting, and the resulting lattice is called the clique-inserted-lattice of  $H$ . For any given  $r$ -regular lattice, applying this operation iteratively, an infinite family of  $r$ -regular lattices is generated. Some interesting lattices including the 3-12-12 lattice can be constructed this way. In this paper, we reveal the relationship between the energy and resistance distance of an  $r$ -regular lattice and that of its clique-inserted-lattice. As an application, the asymptotic energy per vertex and average resistance distance of the 3-12-12 and 3-6-24 lattices are computed. We also give formulae expressing the numbers of spanning trees and dimers of the  $k$ -th iterated clique-inserted lattices in terms of that of the original lattice. Moreover, we show that new families of expander graphs can be constructed from the known ones by clique-inserting.

**Keywords:** Clique-inserting, Lattice, Kirchhoff index, Dimer, Spanning Tree, Expander graph

## 1 Introduction

The study of 2-dimensional lattice models in statistical physics has a long history. One kind of lattices that have been paid a lot of attention to are constructed by replacing each vertex of  $r$ -regular lattices with a complete graph of order  $r$ . (To avoid triviality, we assume  $r \geq 3$  throughout the paper.) Such lattices include the martini [18, 21, 23], the 3-12-12 [19, 21, 22], the 3-6-24 [11] and the modified bath room lattices [21]. Following [29], the operation of transforming each vertex of an  $r$ -regular graph to an  $r$ -clique (complete graph of order  $r$ ) is called *clique-inserting*, and the graph obtained this way is called the clique-inserted-graph of the original graph. From a given  $r$ -lattice, the operation of clique-inserting can also be performed, and the resulting lattice is called the *clique-inserted-lattice* of the original lattice.

Throughout this paper, we always assume that  $G$  denotes an undirected simple graph. Note that in the language of graph theory, the clique-inserting operation on a graph  $G$  can be described as taking the line graph of the subdivision graph of  $G$ . For any given regular lattice, by iterating this operation, a set of hierarchical regular lattices, namely, iterated clique-inserted-lattices can be obtained. For instance, based on the hexagonal lattice, the 3-12-12, 3-6-24 and 3-6-12-48 lattices (refer to [11] for definitions of these lattices) can be generated by clique-inserting. In this case, the clique-inserting operation on a lattice is equivalent to a star-triangle transformation

on the subdivision lattice of the original lattice. By this observation we compute some physical and chemical indices of the 3-12-12 and 3-6-24 lattices from the hexagonal lattice which has been well studied. This approach can also be applied on regular lattices of degree greater than or equal to four, such as the modified bath room lattice (the clique-inserted-graph of square lattice).

For a regular lattice  $H$ , let  $C(H)$  denote the clique-inserted-lattice of  $H$ . Denote the sequence of lattices obtained by iterating clique-inserting operation as follows,  $C^0(H) = H$ ,  $C^1(H) = C(H)$ ,  $C^2(H) = C(C(H))$ ,  $\dots$ ,  $C^k(H) = C(C^{k-1}(H))$ ,  $\dots$  where  $k = 1, 2, \dots$ . In this paper, we consider the lattices produced by the operation of clique-inserting on regular lattices with free, cylindrical and toroidal boundaries. We will discuss the energy per vertex, average resistance (Kirchhoff index over the number of edges) and the entropy of dimer and spanning trees of such lattices. We will also use the operation of clique-inserting to construct new families of expander graphs from known ones.

The dimer problem of lattice has been attracting the attention of many physicists as well as mathematicians. For some classical works, we refer to [6, 20, 22]. Cayley [1] and Kirchhoff [13] presented the problem of enumeration of spanning trees of graphs, and further work in statistical physics has been appearing in both physics and mathematics literature. For a good survey, the reader is referred to [19]. The study of resistance distance is initiated by Klein and Randić [14], and the related index named "Kirchhoff index" was well studied in [10, 24]. Gutman [8] defined the energy of a graph  $G$  as the sum of the absolute values of eigenvalues of  $G$  [9]. Yan and Zhang [27] proposed the energy per vertex problem for lattice systems and showed that the energy per vertex of 2-dimensional lattices is independent of various boundary conditions. For a comprehensive survey of results and common proof methods obtained on graph energy, see [15] and references cited therein. Expander graphs were first defined by Bassalygo and Pinsker in the early 70's. They are regular sparse graphs with strong connectivity properties, measured by vertex, edge or spectral expansion as described in [12]. The property of being such graphs turns out to be significant in various disciplines. For example, expansion is closely related to the convergence rates of Markov Chains and so it plays a crucial role in the study of Monte-Carlo algorithms in statistical mechanics [12]. Due to the usefulness of expander graphs in complexity theory, computer networks and statistical mechanics, their constructions have been studied extensively.

The rest of the paper is organized as follows. In section 2 and 3, the expression of the energy and Kirchhoff index of  $k$ -th iterated clique-inserted-lattices of regular lattices are discussed, respectively. As an application, we compute the energy per vertex and average Kirchhoff index of the 3-12-12 and 3-6-24 lattices. In section 4, we show that, given  $Z_H$  as the entropy of spanning trees of a  $r$ -regular lattice  $H$ , the entropy of spanning trees of  $C^k(H)$  (the  $k$ -th iterated clique-inserted graph of  $H$ ) is given by  $r^{-k}(z_H + s \ln r(r+2))$  where  $s = (r/2 - 1)(r^k - 1)/(r - 1)$ . We will also show that when  $H$  is cubic, the free energy per dimer of the  $C^k(H)$  is  $\frac{1}{3} \ln 2$ . In section 5, we will construct new families of expander graphs from known ones by clique-inserting.

## 2 Asymptotic Energy

Let  $G = (V(G), E(G))$  be a graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E(G)$ . The adjacent matrix of  $G$ , denoted by  $A(G)$ , is an  $n \times n$  symmetric matrix such that  $a_{ij} = 1$

if vertices  $v_i$  and  $v_j$  are adjacent and 0 otherwise. Line graph  $L(G)$  of  $G$ , is a graph such that each vertex of  $L(G)$  represents an edge of  $G$  and two vertices of  $L(G)$  are adjacent if and only if their corresponding edges of  $G$  share a common end vertex in  $G$ . The subdivision graph  $S(G)$  of a graph  $G$  is the graph obtained by inserting a new vertex into every edge of  $G$ . It is easy to see that  $C(G) = L(S(G))$ . The energy of a graph  $G$  with  $n$  vertices, denoted by  $\mathcal{E}(G)$ , is defined by  $\mathcal{E}(G) = \sum_{i=1}^n |\lambda_i(G)|$ , where  $\lambda_i(G)$  are the eigenvalues of the adjacency matrix of  $G$ . The asymptotic energy per vertex of  $G$  [27] is defined by  $\lim_{|V(G)| \rightarrow \infty} \frac{\mathcal{E}(G)}{|V(G)|}$ .

**Lemma 2.1.** [27] Suppose that  $\{G_n\}$  is a sequence of finite simple graphs with bounded average degree such that  $\lim_{n \rightarrow \infty} |V(G_n)| = \infty$  and  $\lim_{n \rightarrow \infty} \frac{\mathcal{E}(G_n)}{|V(G_n)|} = h \neq 0$ . If  $\{G'_n\}$  is a sequence of spanning subgraphs of  $\{G_n\}$  such that  $\lim_{n \rightarrow \infty} \frac{|\{v \in V(G'_n) : d_{G'_n}(v) = d_{G_n}(v)\}|}{|V(G_n)|} = 1$ , then  $\lim_{n \rightarrow \infty} \frac{\mathcal{E}(G'_n)}{|V(G'_n)|} = h$ . That is,  $G_n$  and  $G'_n$  have the same asymptotic energy.

**Lemma 2.2.** [29] Let  $G$  be an  $r$ -regular graph with  $n$  vertices and  $m$  edges. Suppose that the eigenvalues of  $G$  are  $\lambda_1 = r \geq \lambda_2 \geq \dots \geq \lambda_n$ . Then the eigenvalues of the clique-inserted graph  $C(G)$  of  $G$  are  $\frac{r-2 \pm \sqrt{r^2+4(\lambda_i+1)}}{2}$ ,  $i = 1, 2, \dots, n$ , besides  $-2$  and  $0$  each with multiplicity  $m - n$ .

From Lemma 2.2, we immediately obtain the following corollary.

**Corollary 2.3.** Let  $G$  be an  $r$ -regular ( $r \geq 3$ ) graph with  $n$  vertices and eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ , the energy of the clique-inserted graph of  $G$  is

$$\mathcal{E}(C(G)) = \sum_{i=1}^n \frac{r-2 + \sqrt{r^2+4(\lambda_i+1)}}{2} + \sum_{i=1}^n \frac{\sqrt{r^2+4(\lambda_i+1)} - r + 2}{2} + n.$$

We will use this result to calculate the asymptotic energy per vertex of the 3-12-12 lattice and its clique-inserted-lattice in the rest of this section.

## 2.1 3-12-12 lattice

Our notation of hexagonal lattices follows [27]. The hexagonal lattices with toroidal condition, denoted by  $H^t(n, m)$ , are illustrated in Figure 1, where  $(a_1, a_1^*), (a_2, a_2^*), \dots, (a_{m+1}, a_{m+1}^*), (b_1, b_1^*), (b_2, b_2^*), \dots, (b_{n+1}, b_{n+1}^*)$  are edges in  $H^t(n, m)$ .

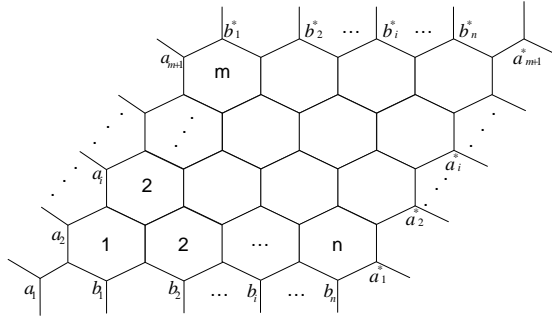


Fig. 1:  $H^t(n, m)$  with toroidal boundary condition

By the definition of clique-inserted-lattice, it is easy to see that each 3-12-12 lattice with toroidal boundary condition is a clique-inserted-graph of  $H^t(n, m)$ , denote as  $T^t(n, m)$  (see Figure 2(a)). Note that  $(a_1, a_1^*), (a_2, a_2^*), \dots, (a_{m+1}, a_{m+1}^*), (b_1, b_1^*), (b_2, b_2^*), \dots, (b_{n+1}, b_{n+1}^*)$  are edges in  $T^t(m, n)$ . If we delete edges  $(b_1, b_1^*), (b_2, b_2^*), \dots, (b_{n+1}, b_{n+1}^*)$  from  $T^t(n, m)$ , then the 3-12-12 lattice with cylindrical boundary condition, denoted by  $T^c(n, m)$  (see Figure 2(b)) can be obtained. If we delete the edges  $(a_1, a_1^*), (a_2, a_2^*), \dots, (a_{m+1}, a_{m+1}^*)$  from  $T^c(m, n)$ , then the 3-12-12 lattice with free boundary condition, denoted by  $T^f(m, n)$  (see Figure 2(c)) can be obtained.

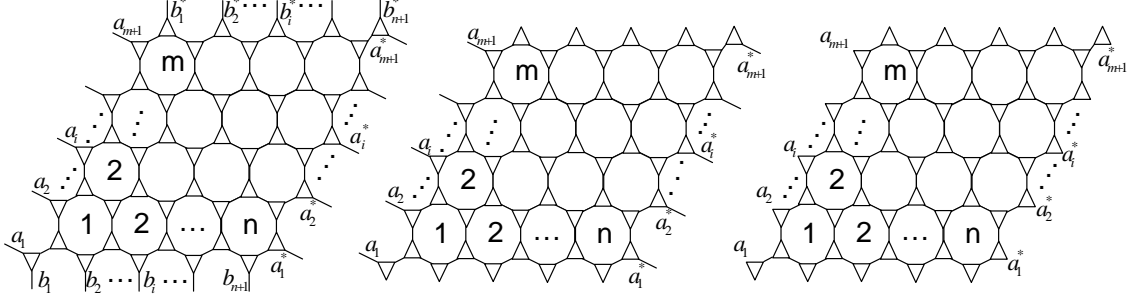


Fig. 2: The 3-12-12 lattice  $T^t(n, m)$  (left),  $T^c(n, m)$  (middle), and  $T^f(n, m)$ .

**Theorem 2.4.** For the 3-12-12 lattices  $T^t(n, m)$ ,  $T^c(n, m)$ , and  $T^f(n, m)$  with toroidal, cylindrical and free boundary conditions,

$$\begin{aligned} \lim_{n, m \rightarrow \infty} \frac{\mathcal{E}(T^t(n, m))}{6mn} &= \lim_{n, m \rightarrow \infty} \frac{\mathcal{E}(T^c(n, m))}{6mn} = \lim_{n, m \rightarrow \infty} \frac{\mathcal{E}(T^f(n, m))}{6mn} \\ &= \frac{1}{3} + \frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} (\sqrt{13 + 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}} \\ &\quad + \sqrt{13 - 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}}) dx dy \approx 1.4825 \end{aligned}$$

that is, the 3-12-12 lattices  $T^t(n, m)$ ,  $T^c(n, m)$ , and  $T^f(n, m)$  with toroidal, cylindrical, and free boundary conditions have the same asymptotic energy ( $\approx 8.895mn$ ).

**Proof.** Note that almost all vertices of  $T^c(m, n)$  and  $T^f(m, n)$  are of degree 3. Since  $T^f(m, n)$  and  $T^c(m, n)$  are spanning subgraphs of  $T^t(m, n)$ , by Lemma 2.1 we have

$$\lim_{n, m \rightarrow \infty} \frac{\mathcal{E}(T^t(n, m))}{6mn} = \lim_{n, m \rightarrow \infty} \frac{\mathcal{E}(T^c(n, m))}{6mn} = \lim_{n, m \rightarrow \infty} \frac{\mathcal{E}(T^f(n, m))}{6mn}$$

It is shown in [27] that the eigenvalues of  $H^t(n, m)$  are:

$$\pm \sqrt{3 + 2\cos \frac{2i\pi}{n+1} + 2\cos \frac{2j\pi}{m+1} + 2\cos \left( \frac{2i\pi}{n+1} + \frac{2j\pi}{m+1} \right)}, 0 \leq i \leq n, 0 \leq j \leq m.$$

Since  $T^t(n, m)$  is the clique-inserted graph of  $H^t(n, m)$ , we have

$$\begin{aligned} \mathcal{E}(T^t(n, m)) &= \sum_{i=0}^n \sum_{j=0}^m \sqrt{13 + 4\sqrt{3 + 2\cos \frac{2i\pi}{n+1} + 2\cos \frac{2j\pi}{m+1} + 2\cos \left( \frac{2i\pi}{n+1} + \frac{2j\pi}{m+1} \right)}} \\ &\quad + \sum_{i=0}^n \sum_{j=0}^m \sqrt{13 - 4\sqrt{3 + 2\cos \frac{2i\pi}{n+1} + 2\cos \frac{2j\pi}{m+1} + 2\cos \left( \frac{2i\pi}{n+1} + \frac{2j\pi}{m+1} \right)}} + 2mn. \end{aligned}$$

Thus, the average energy per vertex of 3-12-12 lattice can be expressed as

$$\lim_{n,m \rightarrow \infty} \frac{\mathcal{E}(T^t(n,m))}{6mn} = \frac{1}{3} + \frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} (\sqrt{13 + 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}} + \sqrt{13 - 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}}) dx dy \approx 1.4825$$

and we complete the proof of the theorem.  $\square$

## 2.2 3-6-24 lattice

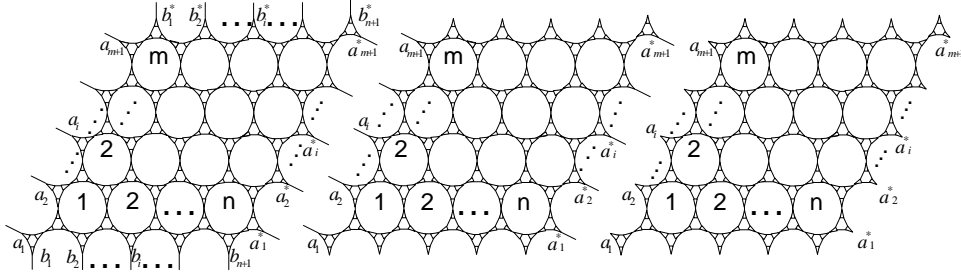


Fig. 3: The 3-6-24 lattice  $S^t(n, m)$  (left),  $S^c(n, m)$  (middle), and  $S^f(n, m)$  (right).

The clique-inserted-graph of  $T^t(m, n)$  is a lattice with toroidal boundary condition, denoted by  $S^t(m, n)$ , illustrated in Figure 3. Note that  $(a_1, a_1^*), (a_2, a_2^*), \dots, (a_{m+1}, a_{m+1}^*), (b_1, b_1^*), (b_2, b_2^*), \dots, (b_{n+1}, b_{n+1}^*)$  are edges in  $S^t(m, n)$ . If we delete edges  $(b_1, b_1^*), (b_2, b_2^*), \dots, (b_{n+1}, b_{n+1}^*)$  from  $S^t(n, m)$ , then the 3-6-24 lattice with cylindrical boundary condition, denoted by  $S^c(n, m)$  (see Figure 3(b)) can be obtained. If we delete edges  $(a_1, a_1^*), (a_2, a_2^*), \dots, (a_{m+1}, a_{m+1}^*)$  from  $S^c(m, n)$ , then the 3-12-12 lattice with free boundary condition, denoted by  $S^f(m, n)$  (see Figure 3(c)) can be obtained.

**Theorem 2.5.** For 3-6-24 lattices  $S^t(m, n)$ ,  $S^c(m, n)$ , and  $S^f(m, n)$  with toroidal, cylindrical and free boundary conditions,

$$\begin{aligned} \lim_{n,m \rightarrow \infty} \frac{\mathcal{E}(S^t(m, n))}{18mn} &= \lim_{n,m \rightarrow \infty} \frac{\mathcal{E}(S^c(n, m))}{18mn} = \lim_{n,m \rightarrow \infty} \frac{\mathcal{E}(S^f(n, m))}{18mn} \\ &= \frac{1}{72\pi^2} \int_0^{2\pi} \int_0^{2\pi} (\sqrt{15 + 2\sqrt{13 + 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}}} \\ &\quad + \sqrt{15 + 2\sqrt{13 - 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}}} \\ &\quad + \sqrt{15 - 2\sqrt{13 + 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}}} \\ &\quad + \sqrt{15 - 2\sqrt{13 - 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}}}) dx dy + \frac{\sqrt{5} + \sqrt{13} + 6}{18} \\ &\approx 1.4908 \end{aligned}$$

that is, the lattices  $S^t(n, m)$ ,  $S^c(n, m)$ , and  $S^f(n, m)$  with toroidal, cylindrical, and free boundary conditions have the same asymptotic energy ( $\approx 26.8344mn$ ).

**Proof.** Note that almost all vertices of  $S^c(m, n)$  and  $S^f(m, n)$  are of degree 3. Since  $S^f(m, n)$  and  $S^c(m, n)$  are spanning subgraphs of  $S^t(m, n)$ , by Lemma 2.1 we have

$$\lim_{n, m \rightarrow \infty} \frac{\mathcal{E}(S^t(n, m))}{18mn} = \lim_{n, m \rightarrow \infty} \frac{\mathcal{E}(S^c(n, m))}{18mn} = \lim_{n, m \rightarrow \infty} \frac{\mathcal{E}(S^f(n, m))}{18mn}$$

The energy of the the clique-inserted-graph of 3-12-12 lattice can be obtained by

$$\begin{aligned} \mathcal{E}(S^t(n, m)) &= \sum_{i=0}^n \sum_{j=0}^m \sqrt{15 + 2\sqrt{13 + 4\sqrt{3 + 4\cos\frac{2i\pi}{n+1} + 2\cos\frac{2j\pi}{m+1} + 2\cos\left(\frac{2i\pi}{n+1} + \frac{2j\pi}{m+1}\right)}}} \\ &+ \sum_{i=0}^n \sum_{j=0}^m \sqrt{15 + 2\sqrt{13 - 4\sqrt{3 + 4\cos\frac{2i\pi}{n+1} + 2\cos\frac{2j\pi}{m+1} + 2\cos\left(\frac{2i\pi}{n+1} + \frac{2j\pi}{m+1}\right)}}} \\ &+ \sum_{i=0}^n \sum_{j=0}^m \sqrt{15 - 2\sqrt{13 + 4\sqrt{3 + 4\cos\frac{2i\pi}{n+1} + 2\cos\frac{2j\pi}{m+1} + 2\cos\left(\frac{2i\pi}{n+1} + \frac{2j\pi}{m+1}\right)}}} \\ &+ \sum_{i=0}^n \sum_{j=0}^m \sqrt{15 - 2\sqrt{13 - 4\sqrt{3 + 4\cos\frac{2i\pi}{n+1} + 2\cos\frac{2j\pi}{m+1} + 2\cos\left(\frac{2i\pi}{n+1} + \frac{2j\pi}{m+1}\right)}}} \\ &+ \sqrt{5}mn + \sqrt{13}mn + 6mn. \end{aligned}$$

The average energy per vertex of the clique-inserted-lattice of the 3-12-12 lattice can be obtained by

$$\begin{aligned} \lim_{n, m \rightarrow \infty} \frac{\mathcal{E}(S^t(n, m))}{18mn} &= \frac{1}{72\pi^2} \int_0^{2\pi} \int_0^{2\pi} (\sqrt{15 + 2\sqrt{13 + 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}}} \\ &+ \sqrt{15 + 2\sqrt{13 - 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}}} \\ &+ \sqrt{15 - 2\sqrt{13 + 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}}} \\ &+ \sqrt{15 - 2\sqrt{13 - 4\sqrt{3 + 2\cos x + 2\cos y + 2\cos(x+y)}}}) dx dy + \frac{\sqrt{5} + \sqrt{13} + 6}{18} \\ &\approx 1.4908 \end{aligned}$$

and we finish the proof of the theorem.  $\square$

### 3 Average Resistance

Kirchhoff index  $Kf(G)$  of a graph  $G$  is defined as the sum of resistance distance between all pairs of vertices of  $G$ . That is,  $Kf(G) = \sum_{\{u,v\} \subset V(G)} R(u, v)$ , where  $R(u, v)$  denotes the resistance

distance between vertices  $u$  and  $v$  of graph  $G$ . Let  $\overline{Kf}(C(G)) = \frac{1}{\binom{n}{2}} Kf(G)$  denote the average

Kirchhoff index, that is, the average resistance distance between all pairs of vertices of  $G$ . Unlike the case of energy per vertex, an edge deletion of the graph may change the average resistance dramatically. For example, if we delete a cut edge of the graph, the resistance distance will become infinity. Thus we only consider lattices with toroidal boundary condition here.

**Lemma 3.1.** [7] *Let  $G$  be a connected  $r$ -regular graph with  $n \geq 2$  vertices. Then*

$$Kf(L(G)) = \frac{r}{2}Kf(G) + \frac{(r-2)n^2}{8}.$$

**Lemma 3.2.** [7] *Let  $G$  be a connected  $r$ -regular graph with  $n \geq 2$  vertices. Then*

$$Kf(s(G)) = \frac{(r+2)^2}{2}Kf(G) + \frac{(r^2-4)n^2+4n}{8}.$$

Combine above two lemmas, the following results are straightforward.

**Proposition 3.3.** *Let  $G$  be a connected  $r$ -regular graph with  $n \geq 2$  vertices. Then*

$$Kf(C(G)) = \frac{r(r+2)^2}{4}Kf(G) + \frac{r^3n^2 - 2rn^2 + 4rn - 4n^2}{16}$$

**Corollary 3.4.** *Let  $G$  be a connected 3-regular graph with  $n \geq 2$  vertices. Then*

$$Kf(C(G)) = \frac{75}{4}Kf(G) + \frac{17n^2 + 12n}{16}$$

### 3.1 3-12-12 lattice

**Theorem 3.5.** *For the 3-12-12 lattice  $T^t(m, n)$  with toroidal boundary condition,*

$$Kf(T^t(m, n)) \approx 204.9788 + \frac{17(m+1)^2(n+1)^2 + 6(m+1)(n+1)}{4},$$

$$\lim_{n, m \rightarrow \infty} \overline{Kf}(T^t(m, n)) = 0.2361111....$$

**Proof.** It is shown in [28] that for the hexagonal lattice  $H^t(m, n)$  with toroidal boundary condition,

$$Kf(H^t(m, n)) \approx 10.9322(m+1)^2(n+1)^2.$$

Therefore, by Corollary 3.4, we have

$$Kf(T^t(m, n)) = 204.9788 + \frac{17(m+1)^2(n+1)^2 + 6(m+1)(n+1)}{4}$$

Thus, the average Kirchhoff index of  $T^t(m, n)$  is given as follows,

$$\lim_{n, m \rightarrow \infty} \overline{Kf}(T^t(m, n)) = \lim_{n, m \rightarrow \infty} \frac{Kf(T^t(m, n))}{\binom{6(m+1)(n+1)}{2}} = 0.2361111...$$

□

### 3.2 3-6-24 lattice

**Theorem 3.6.** *For the 3-6-24 lattice  $S^t(m, n)$  with toroidal boundary condition,*

$$Kf(S^t(m, n)) \approx 3843.3525 + \frac{1887(m+1)^2(n+1)^2 + 522(m+1)(n+1)}{16},$$

$$\overline{Kf}(S^t(m, n)) = 0.7280093....$$

**Proof.** By Corollary 3.4 and Theorem 3.5, we have

$$Kf(S^t(m, n)) \approx 3843.3525 + \frac{1887(m+1)^2(n+1)^2 + 522(m+1)(n+1)}{16}.$$

Thus, the average Kirchhoff index of  $S^t(m, n)$  is given as follows,

$$\lim_{n, m \rightarrow \infty} \overline{Kf}(S^t(m, n)) = \lim_{n, m \rightarrow \infty} \frac{Kf(S^t(m, n))}{\binom{18(m+1)(n+1)}{2}} = 0.7280093....$$

□

## 4 Spanning Tree and Dimer

### 4.1 Spanning Tree

Let  $N_{ST}(G)$  denote the number of spanning trees on  $G$ . It is known that  $N_{ST}(G)$  has asymptotic exponential growth. Define the quantity  $z_G$  by

$$z_G = \lim_{n \rightarrow \infty} \frac{1}{n} \ln N_{ST}(G)$$

This quantity is known as bulk, or the thermodynamic, limit, and we denote  $z_G$  by  $z_L$  for lattices. The following lemma indicates the relation between the number of spanning trees of a regular lattice and that of its  $k$ -th iterated clique-inserted-lattice.

**Lemma 4.1.** [25] *Let  $G$  be an  $r$ -regular graph with  $n$  vertices. Then the number of spanning trees of the iterated clique-inserted-graphs  $C^k(G)$  of  $G$  can be expressed by  $N_{ST}(C^k(G)) = r^{ns-k}(r+2)^{ns+k}N_{ST}(G)$  where  $s = (r/2 - 1)(r^k - 1)/(r - 1)$ .*

Therefore, we obtain the following result.

**Proposition 4.2.** *Let  $H$  be an  $r$ -regular lattice. For  $C^k(H)$  ( $k = 0, 1, 2, \dots$ ), the rate of growth of number of spanning trees,  $z_{C^k(H)}$  is given by  $r^{-k}(z_H + s \ln r(r+2))$  where  $s = (r/2 - 1)(r^k - 1)/(r - 1)$  and  $z_H$  denotes the rate of growth of spanning trees of  $H$ .*

The next Theorem implies that the boundary condition does not affect the bulk limit of a lattice.



**Theorem 4.3.** [17] Let  $\langle G_n \rangle$  be a tight sequence of finite connected graphs with bounded average degree such that

$$\lim_{n \rightarrow \infty} |V(G_n)|^{-1} |\{x \in V(G'_n); \deg_{G'_n}(x) = \deg_{G_n}(x)\}| = 1,$$

then  $\lim_{n \rightarrow \infty} |V(G_n)|^{-1} \log \tau(G'_n) = h$ .

For hexagonal lattice,  $z_{hc}$  is 0.8076649... as shown in [19]. Thus, by Theorem 4.2 and Theorem 4.3, we have that for 3-12-12 lattice and the 3-6-24 lattice with toroidal, cylindrical and free boundary condition,

$$z_{3-12-12} = 0.7205633...$$

$$z_{3-6-24} = 0.6915295....$$

## 4.2 Dimer Problem

Let  $M(G)$  denote the number of dimers (perfect matchings) of  $G$ . The free energy per dimer of  $G$ , denoted by  $Z_G$ , is defined as  $Z_G = \lim_{n \rightarrow \infty} \frac{2}{n} \ln M(G)$ . Given the number of vertices and edges of a connected graph, the number of dimers of the graph and that of its line graph have the following relation.

**Lemma 4.4.** [5] Let  $G$  be a 2-connected graph of order  $n$  and size  $m$ , where  $m$  is even. Then  $M(L(G)) \geq 2^{m-n+1}$ , where the equality holds if and only if  $\Delta(G) \leq 3$ .

With this general result, we can readily obtain the following.

**Theorem 4.5.** Let  $H$  be a cubic lattice  $H$  with toroidal condition, the free energy per dimer of  $C^k(H)$  ( $k = 1, 2, 3, \dots$ ) is  $\frac{1}{3} \ln 2$ .

**Proof.** Let  $H$  be a cubic lattice with  $n$  vertices, then the edge set of  $H$  is of order  $\frac{3n}{2}$ . Note that  $C^{k-1}(H)$  has  $3^{k-1}n$  vertices and  $\frac{3^k n}{2}$  edges and the subdivision of  $C^{k-1}$  has  $\frac{5}{6} \cdot 3^{k-1}n$  vertices and  $3^k n$  edges. Since  $C^k(H)$  is the line graph of the subdivision of  $C^{k-1}(H)$ , by Lemma 4.4 we have  $Z_{C^k}(G) = \lim_{n \rightarrow \infty} \frac{2}{3^k n} \ln 2^{3^k n - \frac{5}{6} \cdot 3^k n + 1} = \frac{1}{3} \ln 2$ .  $\square$

**Theorem 4.6.** Let  $R^t(m, n)$  be the  $k$ -th iterated clique-inserted-lattice of the hexagonal lattice  $H^t(m, n)$  with toroidal boundary. For the lattices  $R^t(m, n)$ ,  $R^c(m, n)$  and  $R^f(m, n)$  with toroidal, cylinder and free boundary conditions,  $Z_{R^t(m, n)} = Z_{R^c(m, n)} = Z_{R^f(m, n)} = \frac{1}{3} \ln 2$ .

**Proof.** Note that  $R^c(m, n)$  ( $R^f(m, n)$ ) can be considered as the line graph of a graph which differs from  $S(C^{k-1}(H^t m, n))$  by small number (small in the sense that the number is  $o(mn)$  as  $m, n$  approach infinity) of edges. Therefore, the results follows by applying Lemma 4.4.  $\square$

In general, when a cubic lattice is a line graph, the free energy per dimer of plane lattices are the same as that of the corresponding cylindrical and toroidal lattices. However, this may not be true when a cubic lattice is not a line graph. The hexagonal lattice is such a counterexample as shown in [26].

## 5 Expander graphs

Let  $d_G(v_i)$  be the degree of vertex  $v_i$  of  $G$  and  $D(G) = \text{diag}(d_G(v_1), d_G(v_2), \dots, d_G(v_n))$  be the diagonal matrix of vertex degree of  $G$ . The Laplacian matrix of graph  $G$  is  $L(G) = D(G) - A(G)$ . The eigenvalues of  $L(G)$ , denoted by  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$  are called the Laplacian spectrum of  $G$ . It is well known that  $\mu_2$  is called the *spectral gap* (or *algebraic connectivity*) of  $G$ , and  $\mu_2$  is greater than 0 if and only if  $G$  is a connected graph. Furthermore,  $\mu_2(G)$  is bounded above by the vertex connectivity of  $G$ . Here we use the spectral gap to quantify the expansion property, that is, a family of regular graphs is an expander family if and only if it has a positive lower bound for the spectral gap, and the larger the bound the better the expansion. This characterization can be formulated to a formal definition as the following.

An infinite family of regular graphs,  $G_1, G_2, G_3, \dots$ , is called a family of  $\epsilon$ -expander graphs [12], where  $\epsilon > 0$  is a fixed constant, if (i) all these graphs are  $k$ -regular for a fixed integer  $k \geq 3$ ; (ii)  $\mu_2 \geq \epsilon$  for  $i = 1, 2, 3, \dots$ ; and (iii)  $n_i = |V(G_i)| \rightarrow \infty$  as  $i \rightarrow \infty$ . Note that for regular graphs,  $\mu_i = r - \lambda_i$  for  $i = 1, 2, \dots, n$ . Then Lemma 2.2 implies that

$$\mu_2(C(G)) = \frac{r + 2 - \sqrt{(r+2)^2 - 4\mu_2(G)}}{2}.$$

Denote the function iteration of  $f(x) = \frac{r+2-\sqrt{(r+2)^2-4x}}{2}$  by  $f^1(x) = f(x)$  and  $f^{k+1}(x) = f(f^k(x))$  for  $k = 1, 2, 3, \dots$ . Then we can readily obtain the following result.

**Corollary 5.1.** *Suppose  $G_1, G_2, G_3, \dots$ , is a family of  $r$ -regular  $\epsilon$ -expander graphs. Then  $C^k(G_1), C^k(G_2), C^k(G_3), \dots$ , is a family of  $r$ -regular  $f^k(\epsilon)$ -expander graphs.*

Lattices with good expansion property can be used to design the network with optimal synchronizability and fast random walk spreading [3]. Though the best expansion for combinatorial expanders is still unknown [16], there is an upper bound on the spectral gap. Given by the spectral gap of the Bethe lattice, Alon and Boppana derived the asymptotic upper bound as follows.

**Theorem 5.2.** [16] *Let  $G_n$  be a family of  $r$ -regular graphs, where  $r$  is fixed and  $n$  is the number of vertices of the graphs. Then*

$$\lim_{n \rightarrow \infty} \mu_2(G_n) \leq r - 2\sqrt{r-1}.$$

A  $r$ -regular graph has  $\mu_2 \geq r - 2\sqrt{r-1}$  (or equivalently say,  $\lambda_2 \leq 2\sqrt{r-1}$ ) is called a *Ramanujan graph*. By Alon-Boppana Theorem, Ramanujan graphs are a family of expanders which achieve the largest possible spectral gap and thus have the best possible expansion from a spectral point of view.

In order to introduce some existing families of Ramanujan graphs and demonstrate our construction of new expander families based on that. The following definitions need to be introduced.

**Definition 5.3.** *Given a group  $\Gamma$  and a generating set  $S \subset \Gamma \setminus \{1\}$  such that  $S^{-1} = S$ , the Cayley graph  $\text{Cay}(\Gamma, S)$  is defined to have vertex set  $X$  such that  $x, y \in \Gamma$  are adjacent if and only if  $x^{-1}y \in S$ .*

**Definition 5.4.** The group  $PGL(2, \mathbb{Z}_q)$  is the quotient group of all two by two invertible matrix over  $\mathbb{Z}_q$  modulo  $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \neq 0 \right\}$

**Definition 5.5.** The group  $PGL(2, \mathbb{Z}_q)$  is the quotient group of all two by two invertible matrix over  $\mathbb{Z}_q$  with determinant one, modulo  $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ .

Given two distinct odd prime number  $p$  and  $q$  such that  $q > 2\sqrt{2}$ . Define the generating set to be

$$S = \begin{pmatrix} a_0 + a_1x + a_3y & -a_1y + a_2 + a_3x \\ -a_1y - a_2 + a_3x & a_0 - a_1x - a_3y \end{pmatrix} \quad (5.1)$$

that satisfies the following

- (i)  $x, y$  are odd primes satisfying  $(x^2 + y^2 + 1) \equiv 0 \pmod{q}$ .
- (ii)  $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$  where  $a_i \in \mathbb{Z}$  and  $a_0$  is a positive odd number if  $p \equiv 1 \pmod{4}$  while if  $p \equiv 3 \pmod{4}$  then  $a_0$  is even and the first non-zero  $a_i$  is positive.

Define  $X^{p,q}$  to be the Cayley graph  $Cay(PSL(2, \mathbb{Z}_q), S)$  when  $p$  is a square module  $q$  and  $Cay(PGL(2, \mathbb{Z}_q), S)$  otherwise. Lubotzky, Phillips and Sarnak provided the above construction and proved that  $X^{p,q}$  is a family of Ramanujan graphs of degree  $p + 1$  for each prime odd  $p$  by using some deep result in number theory [4]. Chiu [2] extended this approach to the case of  $p = 2$  by considering the generating set of

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 + \sqrt{-2} & \sqrt{26} \\ \sqrt{-26} & 2 - \sqrt{-2} \end{pmatrix} \begin{pmatrix} 2 - \sqrt{-2} & -\sqrt{-26} \\ -\sqrt{-26} & 2 + \sqrt{-2} \end{pmatrix} \right\}. \quad (5.2)$$

By acting this generating set on the  $PGL(2, \mathbb{Z}_q)$  or  $PSL(2, \mathbb{Z}_q)$  as before, a family of cubic Ramanujan graphs  $X^{2,q}$  is explicitly constructed.

It is known that [4], for a fixed real number  $0 < \gamma < 1/6$  and sufficiently large  $q$ , the spectral

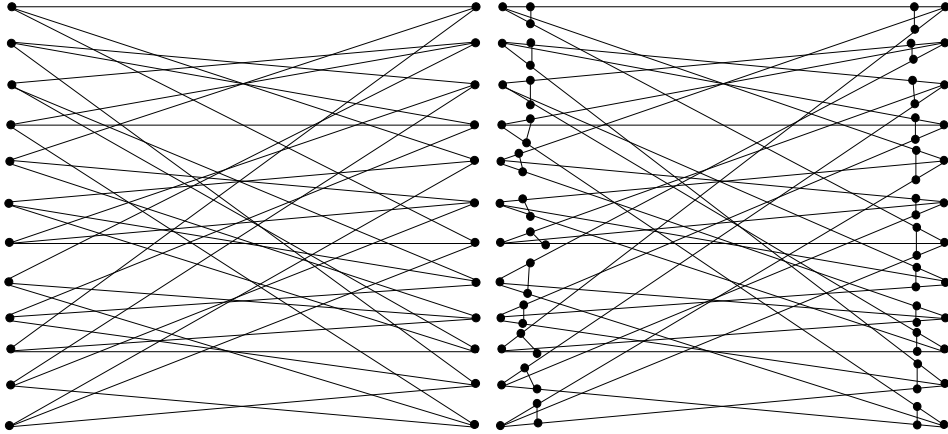


Fig. 4: Cubic Ramanujan graph  $X^{2,3}$  (left) and  $C(X^{2,3})$  (right).

gap of  $X^{p,q}$  is lower bounded by  $\varepsilon(r) = (p + 1) - p^{\frac{5}{6}+\gamma} - p^{\frac{1}{6}-\gamma}$ . Therefore, for a fixed odd prime  $p$ ,  $C(X^{p,q})$  is a  $\frac{p+3-\sqrt{(p+3)^2-4((p+1)-p^{\frac{5}{6}+\gamma}-p^{\frac{1}{6}-\gamma})}}{2}$ -expander family with degree  $p + 1$ . Generally,  $C^k(X^{p,q})$  is a  $f^k((p + 1) - p^{\frac{5}{6}+\gamma} - p^{\frac{1}{6}-\gamma})$ -expander family. Similarly, as we apply clique-inserting iteratively on the expander family  $X^{2,q}$ , new families of cubic expander family can be obtained.

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